PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The Internal Flow Problem in Axi-Symmetric Supersonic Flow

J. J. Mahony

Phil. Trans. R. Soc. Lond. A 1958 **251**, 1-21
doi: 10.1098/rsta.1959.0001

THE INTERNAL FLOW PROBLEM IN AXI-SYMMETRIC SUPERSONIC FLOW

By J. J. MAHONY*
Aeronautical Research Laboratories, Melbourne, Australia

(*Communicated by T. M. Cherry, F.R.S.—Received* 29 *October* 1957)

CONTENTS

A scheme of approximate solution is presented for the supersonic flow in a circular duct of slowly varying cross-section for the cases when the conventional linearized theory fails. This happens whenever there are regions in the flow field, termed wave fronts, where the velocity gradients are large in comparison with the variations in velocity. A careful discussion suggests that a valid first approximation may be obtained from a solution of the linearized equations by placing the solution on Mach linees computed from the solution. This is a natural extension of Whitham's method for the external flow problem. However, it does not suffice to use the ordinary solution of the linearized equations as this possesses singularities. It is necessary to obtain a solution of the linearized equations satisfying boundary conditions in which due allowance has been made for the non-parallelism of the Mach lines. Within the accuracy of the approximation, this solution is found to agree with the ordinary solution away from the wave fronts but differs markedly within them. A simple method is obtained for converting the singular portions of the ordinary solution into a form valid within wave fronts.

The problem, studied by Meyer and Ward, of an expansive discontinuity in the slope of the wall of the duct is discussed and the details of the flow are clarified. It is shown that both the velocity and the velocity gradients are finite on the Mach lines where previous theories predicted singularities. Nevertheless, a shock wave is formed in the reflexion of the expansion wave from the axis of the duct, no matter how small the initial disturbance.

1. Introduction

Linearized theory, originally due to von Kármán & Moore (1932) and greatly developed by Ward (1949) among others, is an extremely powerful tool for the calculation of axi-symmetric supersonic flows. This theory, especially when modified to allow for the convergence and divergence of Mach lines (Whitham 1950, 1952) has been completely successful when applied to predict flows around slender bodies of revolution. However, serious difficulties have been encountered when the theory has been applied to flows, such as in ducts and jets, where the axis of symmetry is not excluded from the flow field. Ward (1948)

* Now at Department of Aeronautics, University of Sydney.

and Meyer (1948) have found that the effects of any discontinuity of slope, curvature, etc., of the external boundary are subject to a radial focusing such that the velocity, or appropriate velocity gradient, becomes infinite at the point O on the axis (figure 1). Even more serious is the fact that the solution of the linearized equations possesses a logarithmic infinity which is propagated along the Mach line OB reflected downstream from the singular point O on the axis. Meyer (1948) has shown further that the solution of linearized theory does not represent the first step in a convergent iteration process for the solution of the full equations governing the flow. By making use of characteristic independent variables, Meyer obtained an approximately solution, for the case when the external boundary has a discontinuity of slope, in which the velocity is continuous along the axis. If z is distance along the axis measured downstream from the point O, then the velocity distribution according to Meyer's solution behaved like $z^{\frac{1}{2}}$, whereas in Ward's solution it behaved like $z^{-\frac{1}{2}}$. Meyer's solution predicted a finite velocity on the Mach line OB but the velocity gradient had a logarithmically infinite behaviour there.

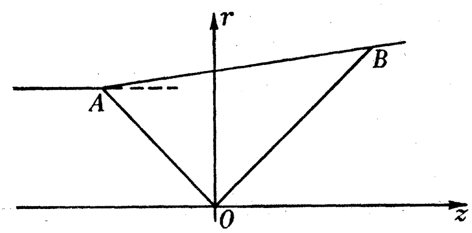

FIGURE 1

Chen (1953) considered the solution of the full equations in the neighbourhood of a point O on the axis at which, for z positive, the expansion for the velocity distribution has a leading term of order $z^{\frac{1}{2}}$. He constructed a convergent iteration for the uniquely determined solution and was able to show that this solution has no logarithmic singularity, in the velocity gradient, propagating along the reflected Mach line. His results are valid only near the origin, but they prove conclusively that Meyer's modification of linearized theory is inadequate. Thus, although Chen's results determine a flow for which the velocity distribution is prescribed on the axis there is no certainty that they apply to the problem Meyer considered, for in order to yield the leading term $z^{\frac{1}{2}}$ in the velocity distribution on the axis Meyer's approximate solution has to be continued across a region where it is not valid. Even so a combination of Meyer's and Chen's solutions would cover only a small neighbourhood of the leading Mach lines AO The solution of linearized theory is available for the remainder of the flow field, but it is not known whether this solution approximates in any way to the solution of the full equations. Certainly it is not valid in the neighbourhood of the singular Mach lines. It is, therefore, of interest to inquire whether the solution of linearized theory is valid in the general body of the flow field away from the wave fronts. In either case, a solution is required in wave fronts in regions away from the axis, which are not covered by Chen's solution.

The problem considered here is that of obtaining an approximate solution for the steady supersonic flow through a duct of circular cross-section and almost constant diameter. Physical experience suggests that, if the flow entering the duct is uniform, the variations of velocity throughout the duct should be small provided no choking effects are encountered.

However, there is no *a priori* reason why the velocity gradients should be small and in fact in both Meyer's and Chen's solutions there are regions where the velocity gradients are large. In such cases the conventional derivation of the linearized equations of motion is not valid. In §2, a set of equations approximating to the full equations is obtained on the assumptions of small variations from a given uniform supersonic velocity and suitable continuity behaviour of the functions describing the velocity field. No restriction is placed on the velocity gradients beyond those implied by the assumed properties of the velocity. This set of approximate equations is closely analogous to and provides a natural extension of that governing Whitham's (1950) version of the linearized theory of external flow valid at large distances. In that superposition of solutions is valid this set of equations is 'effectively linear', but the solutions can have 'non-linear' properties indicative of the presence of shock-waves in the corresponding real flow.

Although it has not been proved here, it seems likely that this set of approximate equations may be used to control a convergent iteration procedure yielding the solution of the full equations when the boundary conditions consist of a prescribed velocity distribution on the axis. It would seem possible to obtain thus the natural extension of Chen's results in regions away from the axis for sufficiently small imposed disturbances. However, of far more practical interest is the behaviour of the solution of the full equations when the boundary conditions are prescribed by specifying the shape of the duct. Thus, in this paper attention will be confined to problems where the boundary conditions imposed are of this latter type. It is hoped that the results obtained by the approximate methods used in this paper will provide the key to the form of the existence theorems for the full equations governing axisymmetric supersonic flow just as Meyer's (1948) results provided the key to Chen's (1953) investigation.

A general solution of the approximate equations is obtained in §3, where the question of finding the solution satisfying the appropriate boundary conditions is discussed. This discussion yields a criterion for the occurrence of wave fronts in the flow field, together with the result that if there are no wave fronts conventional linearized theory provides an adequate description of the flow. When wave fronts are present in the flow field the solution process has to be modified. The results are obtained for a simple typical example for which the complete approximate solution is obtained in §4·2. The particular problem chosen is that of a uniform supersonic jet expanding into a still atmosphere at slightly lower pressure. This problem has the advantage that it is possible to obtain comparatively simple expressions for the solution in the neighbourhoods of the singular Mach lines. Furthermore, the singularities in the velocity gradients are of the worst kind for problems involving wave fronts, so that the *a posteriori* justification of the methods of approximation for this case should apply to all other less singular cases. The properties of this solution are examined at length in §4·3, and it is shown how, because of the effective linearity of the approximate equations, the results may be applied to problems involving more general boundary conditions.

The major results are as follows. Outside wave fronts the solution corresponds with the normal solution of linearized theory, but it may be noted that the typical fractional error involved in the approximation is $\delta \ln \delta$ and not δ as might be suggested by linearized theory.*

* δ is the typical velocity changes imposed at the boundary.

The behaviour of the solution within a wave front and not near the axis may be obtained by a simple operation on the singular portion of the solution of linearized theory. However, within a wave front near the axis, linearized theory fails completely and alternative methods have to be adopted. It is further shown that shock waves must occur in any flow in which the external boundary has a discontinuity of slope. If the discontinuity has the sense of an expansion the shock wave starts with zero strength from the point O (figure 1) on the axis of symmetry through which passes the tail of the centred expansion wave emanating from the point of discontinuity A. In this sense, Chen's result of the existence of a continuous flow field does not apply to the problem originally considered by Ward and Meyer.

2. Basic equations

The equations governing the continuous, irrotational, axi-symmetric supersonic flow of a perfect gas are (see, for example, Howarth 1953): on any line, termed a 'plus' Mach line, which satisfies

$$\mathrm{d}r/\mathrm{d}z = \tan(\theta-\mu), \tag{1}$$

$$\mathrm{d}(\theta+t) = \sin\theta\,\sin\mu\,\mathrm{cosec}\,(\theta-\mu)\,\mathrm{d}r/r. \tag{2}$$

On any line, termed a 'minus' Mach line, which satisfies

$$\mathrm{d}r/\mathrm{d}z = \tan(\theta+\mu), \tag{3}$$

$$\mathrm{d}(\theta-t) = -\sin\theta\,\sin\mu\,\mathrm{cosec}\,(\theta+\mu)\,\mathrm{d}r/r, \tag{4}$$

where r and z are radial and axial position co-ordinates, respectively, θ is the inclination of the flow to the axis and $t(\mu)$ and μ are the local Prandtl and Mach angles, respectively.

Meyer and Chen have shown the great value of using characteristic co-ordinates as independent variables rather than the position co-ordinates r and z. Mahony & Meyer (1956) have shown the value of dealing with the position co-ordinates, which have now to be regarded as dependent variables, via characteristic length parameters. Thus, let ξ and η be a pair of characteristic variables such that a value of ξ defines a particular 'plus' Mach line and a value of η defines a particular 'minus' Mach line. Let the characteristic length parameters U and V be defined by the mapping

$$\mathrm{d}r = f(\mu)\,\{U\sin(\theta+\mu)\,\mathrm{d}\xi + V\sin(\theta-\mu)\,\mathrm{d}\eta\}, \tag{5a}$$

$$\mathrm{d}z = f(\mu)\,\{U\cos(\theta+\mu)\,\mathrm{d}\xi + V\cos(\theta-\mu)\,\mathrm{d}\eta\} \tag{5b}$$

between the characteristic and the flow planes. Here

$$f(\mu) = [\{(\gamma-\cos 2\mu)^{\gamma}\,\mathrm{cosec}^{\gamma+1}\mu\}^{1/(\gamma-1)}\sec\mu]^{\frac{1}{2}}$$

is the function which plays an important role in the theory of two-dimensional supersonic flow and has been tabulated by Mahony & Meyer (1956). Note that, when taken in conjunction with the definitions of ξ and η, equations (5) imply equations (1) and (3).

In the theory of two-dimensional supersonic flow, if the formal solution of the characteristic length parameters is such that either changes sign, no physically realizable continuous flow can exist and shock waves appear in the corresponding real flow. A similar result will now be shown to hold for axi-symmetric flow. For, if either U or V change sign across some line in the characteristic plane, the mapping by equations (5) of the neighbourhood of such a line will produce a multiple covering of the flow plane in which, therefore, at least two characteristics of one family pass through any point. Suppose that in such a region, where the characteristics overlap, θ and t are single-valued functions of position.* Then the two or more characteristics of the same family through any point in this region of the flow plane have the same direction and so every characteristic will be an envelope of other characteristics. This implies that at least one of the characteristic length parameters vanishes throughout the region which is multiply mapped. From equations (5) it follows that the whole of this region must map on to a single line in the flow plane, and furthermore the assumption of a single valued velocity field implies that the flow variables, θ and t are functions of a single characteristic variable. Thus, this trivial case excepted,† the vanishing of a characteristic length parameter implies that the velocity field is a multi-valued function of position in the flow plane.

From the fact that the right-hand sides of equations (5) are perfect differentials it follows that

$$U_\eta + U\cot 2\mu(\theta_\eta + t_\eta) = V\operatorname{cosec} 2\mu(\theta_\xi - \mu_\xi) \tag{6a}$$

and

$$V_\xi - V\cot 2\mu(\theta_\xi - t_\xi) = -U\operatorname{cosec} 2\mu(\theta_\eta + \mu_\eta), \tag{6b}$$

where use has been made of the properties of $t(\mu)$ and $f(\mu)$ (see Meyer 1949). In terms of ξ and η the characteristic compatability relations (2) and (4) become

$$\theta_\eta + t_\eta = \sin\theta \sin\mu f(\mu)\, V/r \tag{7a}$$

and

$$\theta_\xi - t_\xi = -\sin\theta\sin\mu f(\mu)\, U/r. \tag{7b}$$

Equations (5), (6) and (7) are the fundamental characteristic form of the non-linear differential equations governing axi-symmetric supersonic flow. They may be integrated formally to yield

$$U = \exp\left\{-\int \cot 2\mu(\theta_\eta + t_\eta)\,\mathrm{d}\eta\right\}\int V\operatorname{cosec} 2\mu(\theta_\xi - \mu_\xi)\exp\left\{\int\cot 2\mu(\theta_\eta + t_\eta)\,\mathrm{d}\eta\right\}\mathrm{d}\eta, \tag{8a}$$

$$V = \exp\left\{\int \cot 2\mu(\theta_\xi - t_\xi)\,\mathrm{d}\xi\right\}\int U\operatorname{cosec} 2\mu(\theta_\eta + \mu_\eta)\exp\left\{-\int\cot 2\mu(\theta_\xi - t_\xi)\,\mathrm{d}\xi\right\}\mathrm{d}\xi, \tag{8b}$$

$$\theta + t = \int \sin\theta\sin\mu f(\mu)\,(V/r)\,\mathrm{d}\eta, \tag{8c}$$

$$\theta - t = -\int \sin\theta\sin\mu f(\mu)\,(U/r)\,\mathrm{d}\xi, \tag{8d}$$

$$r = \int f(\mu)\,\{U\sin(\theta+\mu)\,\mathrm{d}\xi + V\sin(\theta-\mu)\,\mathrm{d}\eta\}, \tag{8e}$$

$$z = \int f(\mu)\,\{U\cos(\theta+\mu)\,\mathrm{d}\xi + V\cos(\theta-\mu)\,\mathrm{d}\eta\}, \tag{8f}$$

where for brevity the arbitrary functions, which result from the integrations and which are to be determined from the boundary conditions are assumed to be implied by the integral signs.

* In contrast to the case of two-dimensional flow, this is apparently possible, for equations (2) and (4) are not integrable so that each Mach line is not associated with a given value of a flow parameter $\theta \pm t$.

† This case could be removed by a suitable redefinition of one of the characteristic variables. However, this is not convenient in the light of the definition to be adopted in this paper, but it does seem probable, though it has not been proved, that this trivial case cannot occur with the definition adopted here.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

In this paper attention will be confined to cases where both the inclination of the wall of the duct to the axis and the overall change of area are small and typified by a small parameter δ. A further restriction is imposed that the variations across the duct of the velocity and the velocity gradient of the supersonic flow entering the duct are also small and typified by the same small parameter. Because of the complexity of the set of equations (8) it has not yet proved possible to obtain complete solutions of them for the boundary conditions under consideration. Thus, recourse must be made to approximate methods of which the most powerful is to replace equations (8) by simpler approximate forms which are more amenable to analysis. To find suitable approximations for the equations it is necessary to have some knowledge of the properties of the solution of the full equations. With this lacking it is usual to make certain assumptions concerning these properties and then verify *a posteriori* that the solution of the approximate equations obtained in this way has the assumed properties. This method, which though plausible and consistent is not conclusive, will be adopted here.

The question arises as to what properties of the solution may be assumed so that the approximate solution so obtained may be applied to the problem of flow in a duct of slowly varying cross-section. Physical experience suggests that the variations of velocity throughout the flow field are small and tend to zero with δ.* Previous treatments such as Ward's have made the further assumption that the velocity gradients are similarly small. This assumption is too drastic for it is not satisfied by either Meyer's or Ward's solution which does not even statisfy the assumption of small velocity variations. It appears to the author that the least restrictive assumptions which will yet reduce equations (8) to a tractable form are as follows:

(1) The flow variables θ and t are continuous functions of the characteristic variables ξ and η.

(2) The magnitude of the velocity variations $|\theta|$ and $|t-t_0|$ are small and bounded by ϵ, a function of δ which tends to zero as δ tends to zero.

(3) The variations of θ and t are of finite oscillation. These assumptions form the basis of the present investigation.

The reason for the present form of the first assumption rather than the more obvious choice that the flow variables are continuous functions of r and z, lies in the fact that the former permits the consideration of flows involving weak shocks without any fundamental modification of the solution method. Furthermore, it should be emphasized that assumption (2) is weaker than the usual assumption of linearized theory for it allows ϵ to be such that ϵ/δ is not bounded as δ tends to zero. In this regard Meyer's solution suggests that ϵ is $O(\delta^{\frac{1}{2}})$ in certain problems of interest. No restrictions are placed on the velocity gradients beyond those implied by the above assumptions concerning the velocity, together with a weak assumption on the singularities which can occur in the expressions for the velocity gradients. This will be discussed when the need has been established.

The above assumptions have some very important implications as regards the velocity derivatives. Thus these may only be large in comparison with ϵ in ranges of ξ and η which are correspondingly small in comparison with unity. Otherwise assumption (2) would not be satisfied. The regions, corresponding to such small ranges of the characteristic variables,

* At least in the cases where there are no shock waves which are focused on the axis.

will be described hereafter as wave fronts. Note that there are only a finite number of wave fronts within the field and that combined they represent only a small portion of the field. Within almost all the field the assumptions of linearized theory are justified. Within the wave fronts the derivatives of θ and t with respect to ξ and η are not even necessarily bounded, but assumption (1) implies that they are integrable.

It is now possible to deduce what equations are satisfied by any approximation to a solution of the full equations (8) with the properties assumed above. Thus, consider the integrating factor which occurs in equation (8 a) and apply the mean value theorem to each portion of the range of integration within which $(\theta_\eta + t_\eta)$ is one-signed. Thus in each subrange

$$\begin{aligned}\exp\left\{-\int \cot 2\mu(\theta_\eta + t_\eta)\,\mathrm{d}\eta\right\} &= \exp\left\{-\cot 2\bar{\mu}\int(\theta_\eta + t_\eta)\,\mathrm{d}\eta\right\}\\ &= \exp\{-\cot 2\bar{\mu}\Delta(\theta+t)\}\\ &= 1 + O(\epsilon),\end{aligned}$$

where $\bar{\mu}$ is some mean value of the Mach angle in the subrange and Δ is an operator denoting the change in the variable over the range of integration. From assumptions (3) it follows that the contributions from all subranges may be summed and hence the integrating factor differs from unity by a uniformly small quantity. Similar arguments may be applied to each of the equations (8), which may be approximated to by the set of equations

$$U = \operatorname{cosec} 2\mu_0 \int V(\theta_\xi - \mu_\xi)\,\mathrm{d}\eta, \tag{9a}$$

$$V = -\operatorname{cosec} 2\mu_0 \int U(\theta_\eta + \mu_\eta)\,\mathrm{d}\xi, \tag{9b}$$

$$\theta + t = f(\mu_0)\sin\mu_0 \int \theta V/r\,\mathrm{d}\eta, \tag{9c}$$

$$\theta - t = -f(\mu_0)\sin\mu_0 \int \theta U/r\,\mathrm{d}\xi, \tag{9d}$$

$$r = f(\mu_0)\sin\mu_0 \int (U\mathrm{d}\xi - V\mathrm{d}\eta), \tag{9e}$$

$$z = f(\mu_0)\cos\mu_0 \int (U\mathrm{d}\xi + V\mathrm{d}\eta). \tag{9f}$$

Before the solution of the above set of equations it is necessary to complete the definitions of ξ and η. Infinitely many sets of characteristic variables are available and to each set there exists a corresponding pair of characteristic length parameters. Any choice is permissible but the work of Chen (1952) shows that the analysis is simplified by the adoption of a set, such that

$$r = \xi + \eta$$

is a valid first approximation throughout the field. Thus, the following definition has been adopted. If $\xi = 0$ is the 'plus' Mach line bearing the first disturbance due to the duct, then on $\xi = 0$

$$V = -\{f(\mu_0)\sin\mu_0\}^{-1} \tag{10a}$$

and

$$\xi + \eta = 0, \quad r = 0, \quad U = -V = U_0 = \{f(\mu_0)\sin\mu_0\}^{-1}. \tag{10b}$$

Note that equations (9 e) and (10 a) imply that r is equal to $(\xi + \eta)$ on $\xi = 0$.

It is now possible to determine completely the behaviour of the leading terms in equations (9). Equations (9*e*), (9*f*) together with (10*b*) imply that $\int U\,\mathrm{d}\xi$ and $\int V\,\mathrm{d}\eta$ are of the same order as the changes of r and z along the appropriate segment of a Mach line and hence are of unit order at most. Application of the mean value theorem to equation (9*a*) shows that the change in U along a 'plus' Mach line is of the same order as the value of $(\theta_\xi-\mu_\xi)$ on that Mach line. Thus the leading term of the solution for U is constant throughout the field save on 'plus' wave fronts. A further application of the mean value theorem thus shows that the leading term of $\int U\,\mathrm{d}\xi$ may be evaluated without considering the variation of U on 'plus' wave fronts. This is so because the larger the value of $(\theta_\xi-\mu_\xi)$ is on a wave front the correspondingly smaller must be the extent of the wave front. Similar results hold for 'minus' wave fronts and hence the leading terms of equations (9) become

$$U=-U_0\operatorname{cosec}2\mu_0\int(\theta_\xi-\mu_\xi)\,\mathrm{d}\eta, \tag{11a}$$

$$V=-U_0\operatorname{cosec}2\mu_0\int(\theta_\eta+\mu_\eta)\,\mathrm{d}\xi, \tag{11b}$$

$$\theta+t=-\int\theta/r\,\mathrm{d}\eta, \tag{11c}$$

$$\theta-t=-\int\theta/r\,\mathrm{d}\xi, \tag{11d}$$

$$r=\xi+\eta, \tag{11e}$$

$$z=\xi-\eta. \tag{11f}$$

It is possible that in certain problems, conditions may be such as to render invalid the application of the mean value theorem as above. A 'minus' wave front might make a non-small contribution to $\int V(\theta_\xi-\mu_\xi)\,\mathrm{d}\eta$ while making a small contribution to both $\int V\mathrm{d}\eta$ and $\int(\theta_\xi-\mu_\xi)\,\mathrm{d}\eta$. The assumption is made here that this does not happen, so it will be necessary to verify that this is so for any particular solution of the approximate equations.

A schematic method of solution for equations (11) will be presented now without regard to the effect of the form of the imposed boundary conditions. A discussion of the modifications to this method to cope with certain types of boundary conditions is left till § 4. It is easily seen that any function pair satisfying

$$\theta_\eta+t_\eta=-\theta/r=-\theta/(\xi+\eta) \tag{12a}$$

and

$$\theta_\xi-t_\xi=-\theta/r=-\theta/(\xi+\eta) \tag{12b}$$

will satisfy the equations (11*c*) and (11*d*). Note that it is not true that these equations, interpreted as equations for the derivatives of θ and t, are a valid approximation to the complete equations, as may easily be seen by comparing equations (12) and (7). Thus (12*a*) is not valid in any 'minus' wave front, and (12*b*) is not valid in any 'plus' wave front. Similarly, any solution pair of the equations

$$\theta_{\xi\eta}+\tfrac{1}{2}(\theta_\xi+\theta_\eta)/(\xi+\eta)-\theta/(\xi+\eta)^2=0 \tag{13a}$$

and

$$t_{\xi\eta}+\tfrac{1}{2}(t_\xi+t_\eta)/(\xi+\eta)=0, \tag{13b}$$

obtained by eliminating the appropriate dependent variable from equations (12), will satisfy equations (11*c*) and (11*d*), provided equations (12) are satisfied on the boundaries. Thus, in principle, it is possible to determine the leading terms of the solution for θ and t from equations (13), which are essentially those of linearized theory, provided that the boundary conditions are prescribed in a suitable form.

From the above solution it might appear that it is unnecesary to obtain the leading terms of the solution for U and V, as these are of interest only in so far as they yield a determination of r and z for which equations (11*e*) and (11*f*) already provide an adequate approximation. However, as later work will show, U and V may play an important role in the application of the boundary conditions so that the question of their determination will be discussed here. Equations (11*a*) and (11*b*) provide a convenient determination of the leading terms of U and V provided suitable approximations are available for $(\theta_\xi - \mu_\xi)$ and $(\theta_\eta + \mu_\eta)$. Note that it is not necessary, for example, that the approximation for θ_ξ used in equation (11*a*) be correct in a 'minus' wave front for it to yield a valid approximation for U. This suggests that a valid approximation for U and V may be calculated from equations (11*a*) and (11*b*) with values of the derivatives of θ and t obtained from the solution of equations (13). This surmise may be verified *a posteriori* by a consideration of, for example, equation (7*b*) and

$$\theta_\xi + t_\xi = \int \frac{\partial}{\partial \xi} [\sin\theta \sin\mu f(\mu)\, V/r]\, d\eta,$$

which has been obtained from equation (7*a*) by differentiation with respect to ξ and integration with respect to η. These may be regarded as a pair of simultaneous linear equations determining θ_ξ and t_ξ and, in all cases considered by the author, it was valid to evaluate $\int \theta_\xi d\eta$, etc., using values of θ_ξ and t_ξ calculated from

$$\theta_\xi - t_\xi = -\theta/(\xi+\eta)$$

and

$$\frac{\partial}{\partial \eta}(\theta_\xi + t_\xi) = -\frac{\partial}{\partial \xi}[\theta/(\xi+\eta)].$$

This is equivalent to using values obtained by differentiating the solutions of equations (12) or (13). Similar results hold for the η derivatives.

It is of interest to interpret the process whereby the leading terms of the solution are obtained. Equations (13) are effectively the equations of linearized theory wherein the Mach lines are treated as parallel. Equations (11*a*) and (11*b*) then displace these Mach lines, in the flow plane. This distribution of the solution of the linearized equations along Mach lines calculated from it, rather than on the parallel Mach lines of linearized theory itself, is the essence of Whitham's (1950, 1952) treatment of the external flow. In his case it is necessary to consider the displacement of only one family of Mach lines, but in the present case it is necessary to displace both families.

Throughout this paper interest is centred on obtaining approximations to the solution of the full equations rather than establishing results concerning the solution. Nevertheless, it is worthwhile to consider the implications of the above results as regards the complete solution. One possible line of approach to this problem is to try to establish a convergent iteration for sufficiently small variations of velocity throughout the field. The above results show that it would not be possible to establish a convergent iteration based on improving an approximation by inserting it in the right-hand sides of equations (8). For the form of the solution for the leading term shows that an approximation would not be improved at each successive stage unless the leading terms of the last correction satisfied equations (11). Therefore, to have any chance of a convergent iteration it would be necessary to replace equations (8c) and (8d) by

$$\theta_\eta + t_\eta + \theta/(\xi+\eta) = \{\sin\theta \sin\mu f(\mu)\, V/r + \theta(/\xi+\eta)\}$$

and

$$\theta_\xi - t_\xi + \theta/(\xi+\eta) = \{\sin\theta \sin\mu f(\mu)\, U/r + \theta/(\xi+\eta)\}.$$

In essence this is the basis which Chen used to obtain an iteration which converged in a small neighbourhood of the axis. The present results suggest, therefore, that Chen's existence theory can be extended to cover the case of a non-small domain provided that the variations of velocity along the axis of symmetry within the domain of dependence of the solution are sufficiently small. The effort involved in such an extension would be large and the results would be restricted to the case where the velocity distribution is prescribed on the axis. It was thought more profitable to consider the behaviour of the approximate solution when the boundary conditions are prescribed on the wall of the duct in the hope that this may lead to the form of existence theorems, which are as yet unknown, for such boundary conditions as normally occur in engineering problems.

3. The general solution

Operational methods may be applied to obtain the general solutions of equations (13) and also to obtain certain special forms of solution. An alternative form of equations (13),

$$\theta_{yy} - \theta_{xx} + \theta_y/y - \theta/y^2 = 0 \tag{14a}$$

and

$$t_{yy} - t_{xx} + t_y/y = 0, \tag{14b}$$

suitable for the application of operational methods, may be obtained by transforming to the new variables

$$y = \xi+\eta, \quad x = 1+\xi-\eta, \tag{15}$$

which are identical with the first approximations for r and $z\cot\mu_0$. If $x = 0$ is chosen within a region of uniform supersonic flow with $t = t_0$ the Laplace transforms Θ and T of θ and $t-t_0$ satisfy the pair of ordinary differential equations

$$\frac{d^2\Theta}{dy^2} + 1/y\frac{d\Theta}{dy} - (p^2+1/y^2)\,\Theta = 0$$

and

$$\frac{d^2T}{dy^2} + 1/y\frac{dT}{dy} - p^2T = 0.$$

These are forms of Bessel's equations and the solutions, which are bounded on the axis of symmetry, are

$$\Theta = A(p)\,I_1(py),$$

$$T = B(p)\,I_0(py),$$

where $A(p)$ and $B(p)$ are arbitrary functions of p. In order that solutions of equations (13) may satisfy equations (11c) and (11d), it is necessary to satisfy equations (12), which may be written

$$\theta_x = t_y \quad \text{or} \quad p\Theta = dT/dy$$

and

$$t_x = \theta_y + \theta/y \quad \text{or} \quad pT = d\Theta/dy + \Theta/y.$$

It follows therefore, that in the present problem,

$$A(p) = B(p).$$

Hence, the general solutions for the leading approximations for θ and t are

$$\theta(x, y) = (2\pi \mathrm{i})^{-1} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} A(p)\, I_1(py)\, \mathrm{e}^{px} \mathrm{d}p \tag{16a}$$

and

$$t(x, y) - t_0 = (2\pi \mathrm{i})^{-1} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} A(p)\, I_0(py)\, \mathrm{e}^{px} \mathrm{d}p. \tag{16b}$$

Two interpretations of $A(p)$, which are of value in subsequent work, are readily obtainable. Thus from (16*b*)

$$t(x, 0) = t_0 + (2\pi \mathrm{i})^{-1} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} A(p)\, \mathrm{e}^{px}\, \mathrm{d}p,$$

so that if

$$t(x, 0) = t_0 + \mathscr{T}_0(x) \tag{17a}$$

then

$$A(p) = \int_0^\infty \mathscr{T}_0(x)\, \mathrm{e}^{-px}\, \mathrm{d}x. \tag{17b}$$

Alternatively, if

$$t(x, 1) = t_0 + \mathscr{T}_1(x) \tag{18a}$$

then

$$A(p) = \int_0^\infty \mathscr{T}_1(x)\, \mathrm{e}^{-px}\, \mathrm{d}x / I_0(p). \tag{18b}$$

If $A(p)$ is eliminated between equations (16) and (17) by making use of the faltung theorem and known transforms (Erdelyi *et al.* 1954) it can be shown that an alternative form of the general solution is

$$\theta(x, y) = (1/\pi)\, y \int_0^1 \{u(1-u)\}^{\frac{1}{2}} \mathscr{T}_0'[x + y - 2yu]\, \mathrm{d}u \tag{19a}$$

and

$$t(x, y) = t_0 + (1/\pi) \int_0^1 \{u(1-u)\}^{-\frac{1}{2}} \mathscr{T}_0[x + y - 2yu]\, \mathrm{d}u. \tag{19b}$$

From equations (16) it follows that θ and t will exist, be continuous and possess bounded first derivatives throughout the flow field provided that $A(p)$ is $o(p^{-2})$ when p is large, for then the integrals are uniformly convergent. This condition is satisfied when the Laplace transform of $\mathscr{T}_1(x)$ is $o(p^{-\frac{5}{2}})$, which is so if $\mathscr{T}_1(x), \mathscr{T}_1'(x)$ exist and are continuous and $\mathscr{T}_1''(x)$ has at worst isolated singularities of the form $(x - x_i)^n$ with $n > -\frac{1}{2}$. Similar results may be obtained in terms of the behaviour of $\theta = \theta_1(x)$ on $y = r = 1$.

Consider the problem of determining the nearly uniform supersonic flow in an axi-symmetric duct or jet in the cases when either the shape of the boundary or the pressure distribution thereon is prescribed. These are equivalent to the cases when θ and $t - t_0$, respectively, are prescribed on the boundary. Since the percentage variation in the diameter of the duct or jet is small, the leading approximation to the solution may be obtained by applying the boundary conditions on $r = 1$* so that either $\mathscr{T}_1(x)$ or $\theta_1(x)$ is equal to a known function $F(x, \delta)$ which is uniformly $O(\delta)$. Consider the above solution (18) when either θ_1 or $\mathscr{T}_1$ takes the value $\delta F_0(x)$, where F_0 is the uniformly bounded function defined by

$$F_0(x) = \lim_{\delta \to 0} F(x, \delta)/\delta. \tag{20}$$

* The unit of length has been chosen so that the mean radius of the duct is 1.

This solution will exist and be continuous, with bounded first derivatives, under the conditions stated above and furthermore it will be a uniformly continuous function of δ. Under these conditions the solution so obtained should approximate the solution of the full equations with boundary conditions θ_1 or $\mathscr{T}_1$ equal to $F(x, \delta)$. There are no wave fronts in such a problem and the approximate solution obtained above is identical with the solution of linearized theory.

When $F_0(x)$ does not satisfy these conditions wave fronts will occur in the flow field and it is necessary to modify the above solution according to the nature of the singularity. The remainder of this paper will be devoted to a discussion of the worst possible singularity—namely, when F_0 is discontinuous. It is worth noting that this singularity—or any other—can arise in two ways. The discontinuity of slope or pressure may occur in the original boundary condition, or it may arise from the limiting process of equation (20). The effects are the same although the two cases require slightly different treatments. For a discussion of the second case the reader is referred to the work of Mahony (1957). The case where the original boundary conditions possess a discontinuity is discussed in the next two sections, at first for a special simple example which serves to establish the features of the solution. It is then easy to show how the results may be extended for general boundary conditions.

4. The expanding jet

The simple type problem which yields the key to the treatment of flows involving wave fronts is that of the steady flow in a jet formed by a uniform supersonic stream issuing from a perfect nozzle of circular cross-section through an atmosphere at a slightly lower pressure. Let δ be the increase in Prandtl angle which results as the gas expands isentropically from its initial state in the nozzle through the pressure difference between the gas in the nozzle and the still atmosphere. A centred expansion wave, initially similar to a Prandtl–Meyer fan, starts from the lip of the nozzle and suffers a series of reflexions from the axis and the boundary of the jet as it propagates downstream (figure 2). The boundary conditions for this problem are

(1) uniform incident stream

$$\theta = 0, \quad t = t_0 \quad \text{for} \quad \xi \leqslant 0; \tag{21a}$$

(2) symmetry condition on axis

$$\theta = 0 \quad \text{on} \quad y = \xi + \eta = 0; \tag{21b}$$

(3) locally centred expansion wave at lip of nozzle

$$r = 1, \quad z = 0 \quad \text{on} \quad \eta = 1 \quad (0 \leqslant \xi \leqslant \xi(C) = \alpha);$$

which from equations (5) and (7*b*) may be written

$$U = 0, \quad \theta - t = -t_0 \quad \text{on} \quad \eta = 1 \quad (0 \leqslant \xi \leqslant \alpha); \tag{21c}$$

(4) constant pressure on jet boundary

$$t = t_0 + \delta \quad \text{on} \quad y = \xi + \eta = 1 \quad (\xi \geqslant \alpha). \tag{21d}$$

The difference in inclination of the Mach lines AB and AC (figure 2) is of the same order as the variations of the velocity and hence the distance BC is small. It follows, therefore, from the defined scale of the characteristic length parameters on the axis, that α is small and of the same order as the velocity variations. As the boundary conditions are at present stated there is an insufficiency of boundary conditions to determine θ and t from equations (13) and a corresponding excess of boundary conditions on U and V. It is thus necessary to determine say $t(\xi)$ on $\xi+\eta = 1$* for $0 \leqslant \xi \leqslant \alpha$ so that the solution obtained therefrom satisfies equation (21 c). This of course spoils the schematic solution of § 3.

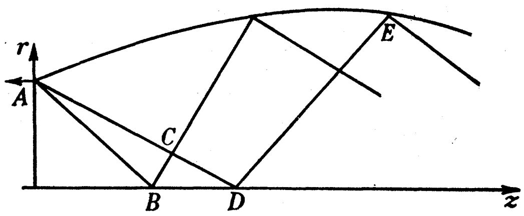

FIGURE 2 (a). Flow plane.

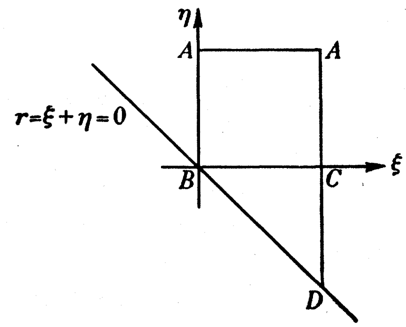

FIGURE 2 (b). Characteristic plane.

4·1. *Conventional linearized theory*

It is stated by Howarth (1953) that Taunt & Ward have obtained the solution of this problem of the expanding supersonic jet using conventional linearized theory but no detailed account has been published. Some of the properties of such a solution will be required later, so a brief outline will be given here. If one applies the limiting process of equation (20) to the boundary conditions, (21 c) disappears and condition (21 d) is replaced by

$$t-t_0+\delta \quad \text{on} \quad y = \xi+\eta = 1 \quad (x>0), \tag{22}$$

which is equivalent to assuming $\alpha = 0$ in the original boundary conditions. As $(t-t_0)$ is less than δ for $\xi<\alpha$ it would seem that this further approximation should involve an error of order $\alpha\delta$ and hence should be negligible for the first-order approximation sought here. Later work will show that the effects of this approximation are not everywhere of second order. If this is so, the reader may query the validity of the original boundary condition (21 d) in which the prescribing of constant pressure on $y = 1$ also involves a second-order error on the boundary. A close examination of the subsequent work reveals that the conventional linearized theory fails because it makes an error of order unity in t_ξ over a range of size α and this leads to a first-order error in a portion of the field. Approximating to the boundary conditions is valid only if the derivatives are also small. In the case of the remainder of the jet boundary† this is so because the slope and curvature of the boundary are of the same order as the velocity variations.

With the boundary condition (22) it may be shown that the solution (16) becomes

$$t(x,y) = t_0+\delta(2\pi \mathrm{i})^{-1}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} p^{-1}\,\mathrm{e}^{px}\,I_0(py)/I_0(p)\,\mathrm{d}p. \tag{23}$$

* Note that this is a sufficient approximation to $\eta = 1$ for the first-order approximation sought here.

† Exception must be made of downstream regions where the reflected wave fronts meet the jet boundary. Each of these special regions may be treated in turn in the same fashion as the first will be treated. As such regions have no upstream influence ignoring their existence does not invalidate the solution obtained upstream of them.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

From the asymptotic behaviour of I_0 for large values of the argument it can be shown that the integral may be evaluated by closing the contour in the right half-plane if $x+y<1$ (i.e. $\xi<0$). Hence

$$t(x,y) = t_0 \quad \text{if} \quad x+y<1.$$

For $x+y>1$ Taint & Ward evaluate the integral by closing the contour in the left half-plane and applying the residue theorem. This yields the infinite series

$$t(x,y) = t_0+\delta-2\delta\sum_{n=1}^{\infty} J_0(\lambda_n y)\cos\lambda_n x/[\lambda_n J_1(\lambda_n)], \tag{24}$$

from which the singularities may be extracted by considering the behaviour of the terms of the series for large n. An example where this method has been applied to a similar series may be found in Ward (1948). However, the method does not yield a complete description of the singularities owing to the necessity of distinguishing between the cases $y=0$ and $y\neq 0$. The following alternative treatment overcomes this difficulty. Any finite portion of the range of integration in equation (23) yields an analytic function of x and y, so that any singularities arise only from the portion of the range where $|p|$ is large when

$$I_0(p) \sim (2\pi p)^{-\frac{1}{2}}\{e^p[1+O(p^{-1})]\pm i\,e^{-p}[1+O(p^{-1})]\},$$

where the $\pm$ signs are to be taken according to the sign of the imaginary part of p. The terms denoted by $O(p^{-1})$ in the above expression will contribute terms which are uniformly $O(p^{-\frac{3}{2}})$ to the integrand and hence will yield a continuous function of x and y. Thus

$$\begin{aligned} t(x,y) &= f_0+[\delta/(2\pi i)]\Big[\int_0^{c+i\infty}(2\pi/p)^{\frac{1}{2}}I_0(py)\,e^{p(x-1)}(1+i\,e^{-2p})^{-1}\,dp \\ &\qquad +\int_{c-i\infty}^{0}(2\pi/p)^{\frac{1}{2}}I_0(py)\,e^{p(x-1)}(1-i\,e^{-2p})^{-1}\,dp+\text{continuous function}\Big] \\ &= t_0+[8/(2\pi i)]\Big[\sum_{n=0}^{\infty}\Big\{(-i)^n\int_0^{c+i\infty}(2\pi/p)^{\frac{1}{2}}I_0(py)\,e^{p(x-1-2n)}\,dp\Big\} \\ &\qquad +(i)^n\int_{c-i\infty}^{0}(2\pi/p)^{\frac{1}{2}}I_0(py)\,e^{p(x-1-2n)}\,dp+\text{continuous function}\Big] \end{aligned} \tag{25}$$

which, for brevity, may be written

$$t = t_0+[8/(2\pi i)]\Big[\sum_{n=0}^{\infty}(-i)^n S_n(X_n,y)+\text{continuous function}\Big], \tag{26a}$$

where

$$X_n = x-(2n+1). \tag{26b}$$

The paths of integration may be deformed to coincide with the positive real axis when $X_n+y<0$ and with the negative real axis when $X_n-y>0$. In the range $|X_n|<y$ the integrals may be transformed to integrals along the real axis by change of variable. Thus it may be shown that

$$S_n(X_n,y) = \begin{cases} (2\pi)^{\frac{1}{2}}\{1-(-1)^n\}\int_0^{\infty}t^{-\frac{1}{2}}I_0(ty)\,e^{tX_n}\,dt & (X_n<-y), \\ (2\pi)^{\frac{1}{2}}e^{i\frac{1}{4}\pi}\int_0^{\infty}t^{-\frac{1}{2}}J_0(ty)\,\{e^{itX_n}+(-1)^n i\,e^{-itX_n}\}\,dt & (|X_n|<y), \\ (2\pi)^{\frac{1}{2}}i\{1+(-1)^n\}\int_0^{\infty}t^{-\frac{1}{2}}I_0(ty)\,e^{-tX_n}\,dt & (X_n>y). \end{cases}$$

These Lipschitz–Hankel integrals may be expressed either in terms of Legendre functions or equivalent hypergeometric functions (see Watson 1944)

$$S_n(X_n, y) = \begin{cases} 2\{1-(-1)^n\} y^{-\frac{1}{2}} Q_{-\frac{1}{2}}(-X_n/y) & (X_n < -y), \\ \pi(y^2 - X_n^2)^{-\frac{1}{4}} F(\frac{1}{4}, \frac{1}{4}, 1, y^2/(y^2-X_n^2))\, e^{i\pi[1+(-1)^n]/4} & (|X_n| < y), \\ 2i\{1+(-1)^n\} y^{-\frac{1}{2}} Q_{-\frac{1}{2}}(X_n/y). \end{cases} \tag{27}$$

From the regularity properties of the Legendre and hypergeometric functions it follows that $S_n(X_n, y)$ is singular only on the lines $X_n \pm y = 0$. Thus it follows from equation (26*a*) that t is discontinuous or singular only on these lines and that the behaviour in the neighbourhood of the singularities may be deduced from equation (27). Furthermore, equations (26) and (27) show that the pattern of the singularities is periodic with a spacing of eight units in the x-direction. Moreover, the pattern is reproduced, with a reversal of sign, at every four units in the x-direction. From equations (27) it is possible to obtain simple expressions for the singularities, equivalent to those obtained by Taunt & Ward by the series method, for points away from the axis by using the leading terms

$$Q_{-\frac{1}{2}}(z) \approx -\tfrac{1}{2}\ln(z-1) + \text{constant},$$

$$F(\tfrac{1}{4}, \tfrac{1}{4}, 1, z) \approx \{\Gamma(\tfrac{1}{4})\,\Gamma(\tfrac{3}{4})\} z^{-\frac{1}{4}}\{\ln z + \text{constant}\}$$

of the expansions of the functions near their singular points (see Erdelyi *et al.* 1953). For a complete discussion of the singularities at points on the axis it is necessary to consider the entire range from 1 to ∞ for the functions in equation (27).

Table 1

singular line	behaviour	nature of disturbance
$y+x = 4m+1$	discontinuity of $\delta y^{-\frac{1}{2}}$	expansion
$y-x = -(4m+1)$	$-(\delta/\pi)\, y^{-\frac{1}{2}} \ln \mid 2 \sin \pi\eta \mid$	expansion followed by compression
$y+x = 4m+2$	$(\delta/\pi)\, y^{-\frac{1}{2}} \ln \mid 2 \sin \pi\xi \mid$	compression followed by expansion
$y-x = -(4m+2)$	discontinuity of $\delta y^{-\frac{1}{2}}$	expansion
$y+x = 4m+3$	discontinuity of $-\delta y^{-\frac{1}{2}}$	compression
$y-x = -(4m+3)$	$(\delta/\pi)\, y^{-\frac{1}{2}} \ln \mid 2 \sin \pi\eta \mid$	compression followed by expansion
$y+x = 4m+4$	$-(\delta/\pi)\, y^{-\frac{1}{2}} \ln \mid 2 \sin \pi\xi \mid$	expansion followed by compression
$y-x = -(4m+4)$	discontinuity of $-\delta y^{-\frac{1}{2}}$	compression

$m = 0, 1, 2, \ldots$.

Note that the existence of compressive discontinuities in this solution does not necessarily imply the presence of the shock waves in the real flow. This can be seen by considering the corresponding problem in two-dimensional flow where, in higher-order theories, the compressive discontinuities are spread out into narrow compression waves (see Mahony & Meyer 1956). There is a tendency towards shock formation within such waves, but the shock wave may not occur within the confines of the flow field. The criterion for shock formation is a change in sign of one of the characteristic length parameters U and V. This question will be examined further when the complete first-order approximation has been obtained.

4·2. *Valid first-order theory*

The infinitely large velocities which occur in the conventional solution of linearized theory obtained in the previous section are not observed in actual flows. Since the differential equations may be expected to yield a satisfactory description of the flow it would appear that the unsatisfactory nature of the above solution is due to placing α equal to zero in the boundary conditions. When α is not zero the schematic solution of § 3 can be applied only when the behaviour of t on $r = 1$ for $0 \leqslant \xi \leqslant \alpha$ is known. To determine this it is necessary to consider only the solution within the centred wave ABD (figure 2a), for the data prescribed on boundaries downstream of the Mach line AD have no influence. Rather than assume some arbitrary variation of t on the boundary to be determined from an integral equation it is convenient to guess the form of the solution from Meyer's (1948) results and then verify that it is correct. Thus, if one assumes that along the axis for ξ negative $(t-t_0)$ vanishes and has the leading term $\kappa\xi^{\frac{1}{2}}$ for ξ positive, the results of § 3 may be applied to yield

$$t(\xi,\eta) = t_0 + (\kappa/\pi)\int_0^L \{[\xi-(\xi+\eta)\,u]/[u(1-u)]\}^{\frac{1}{2}}\,\mathrm{d}u \tag{28a}$$

and

$$\theta(\xi,\eta) = (\kappa/\pi)\,(\xi+\eta)\int_0^L \{u(1-u)/[\xi-(\xi+\eta)\,u]\}^{\frac{1}{2}}\,\mathrm{d}u, \tag{28b}$$

where L is the smaller of 1 and $\xi/(\xi+\eta)$, for the leading terms of the solution for t and θ within the centred wave. The leading term of the solution for the characteristic length parameter U may now be determined from equations (11a) and (11b) and (28). This determination is greatly simplified if one first uses the fact that wave fronts η constant do not contribute to the leading term for U. Hence U may be determined within the centred wave from the approximate forms of equations (28) valid when η is very much larger than ξ. Suitable approximations are

$$\theta \approx t-t_0 \approx (\kappa/\pi)\,\xi(\xi+\eta)^{-\frac{1}{2}}\int_0^1 \{(1-v)/v\}^{\frac{1}{2}}\,\mathrm{d}v$$
$$\approx \tfrac{1}{2}\kappa\xi\eta^{-\frac{1}{2}}\{1+O(\xi/\eta)\}.$$

Equations (10b) and (11a) yield

$$U = U_0\{1-2\kappa m_0\eta^{\frac{1}{2}}\}+O(\alpha^{\frac{1}{2}}) \quad (0\leqslant\xi\leqslant\alpha),$$

where

$$m(\mu) = \frac{1}{2}\left(1-\frac{\mathrm{d}\mu}{\mathrm{d}t}\right)\operatorname{cosec} 2\mu = \tfrac{1}{4}(\gamma+1)\sec^2\mu\operatorname{cosec}2\mu$$

and $m_0 = m(\mu_0)$. Thus the boundary condition (21c) for U will be satisfied if

$$\kappa = 1/(2m_0). \tag{29}$$

The value of t on the boundary of the jet,

$$t(\xi,1-\xi) = t_0 + (2m_0\pi)^{-1}\int_0^\xi \{(\xi-u)/[u(1-u)]\}^{\frac{1}{2}}\,\mathrm{d}u$$
$$= t_0+\xi/(4m_0)+\text{smaller terms} \quad (0\leqslant\xi\leqslant\alpha)$$

may be deduced from equation (28). As t must be continuous across $\xi=\alpha$, equating the values of t on either side of this Mach line yields

$$\alpha = 4m_0\delta.$$

The solution can be completed by verifying that the velocity distribution on the axis does in fact possess a leading term $\kappa\xi^{\frac{1}{2}}$ with appropriate value of κ.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Thus the boundary conditions (21 *c*) and (21 *d*) may be replaced by the equivalent conditions on $\xi+\eta = 1$

$$t = t_0+\xi/(4m_0) \quad (0 \leqslant \xi \leqslant 4m_0\delta),$$
$$t = t_0+\delta \quad (\xi \geqslant 4m_0\delta);$$

or in terms of x and y on $y = 1$

$$t = t_0+x/(8m_0) \quad (0 \leqslant x \leqslant 8m_0\delta),$$
$$t = t_0+\delta \quad (x \geqslant 8m_0\delta).$$

The approximate solution for t may now be obtained from the results of § 3 and is found to be

$$t(x,y) = t_0+(2\pi \mathrm{i})^{-1}(8m_0)^{-1}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} p^{-2}\,\mathrm{e}^{px}\,[1-\mathrm{e}^{-m_0\delta p}]\,I_0(py)/I_0(p)\,\mathrm{d}p,$$

which may be written in the alternative form

$$t(x,y)-t_0 = \{G(x,y)-G(x-8m_0\delta,y)\}/(8m_0), \tag{30a}$$

where
$$G(x,y) = (2\pi \mathrm{i})^{-1}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} p^{-2}\,\mathrm{e}^{px}\,I_0(py)/I_0(p)\,\mathrm{d}p. \tag{31a}$$

Similarly it may be shown that

$$\theta(x,y) = \{H(x,y)-H(x-8m_0\delta,y)\}/(8m_0), \tag{30b}$$

where
$$H(x,y) = (2\pi \mathrm{i})^{-1}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} p^{-2}\,\mathrm{e}^{px}\,I_1(py)/I_0(p)\,\mathrm{d}p. \tag{31b}$$

Since the integrands in equations (31) are $O(p^{-\frac{3}{2}})$ for p large, θ and t as given by equations (30) are continuous functions of x and y. Furthermore, the methods of § 4·1 may be used to show that G and H possess continuous derivatives with respect to x and y everywhere save on those same isolated Mach lines at which the conventional solution of linearized theory is not continuous.

From equation (30) it follows that

$$t(x,y)-t_0 = \delta G_x(x,y)+O(\delta^2) \tag{32a}$$

provided that G_{xx} is bounded in a suitable δ neighbourhood of x. This property may be shown to hold by the methods of the previous section at all points outside a δ-neighbourhood of the singular Mach lines of the previous solution. Now by differentiation of equation (31 *a*) and comparison of the result of equation (23) it can be shown that

$$\delta G_x(x,y) = \mathscr{L}(x,y), \tag{32b}$$

where $\mathscr{L}$ is the conventional solution of linearized theory obtained in § 4·1. Thus $\mathscr{L}$ provides a valid approximation everywhere save in the neighbourhood of the Mach lines on which it is not continuous. When it is a valid approximation the order of difference between $\mathscr{L}$ and the complete solution will be that of the greater of δ^2 and the order of the approximation involved in the solution of equations (30) and (31) Further, the solution of linearized theory may be used to provide a valid approximation even within wave fronts because G_x and $\mathscr{L}$, though singular, are integrable and hence equation (30) may be written in the form

$$t(x,y) = t_0+(8m_0\delta)^{-1}\int_{x-8m_0\delta}^{x}\mathscr{L}(x,y)\,\mathrm{d}x. \tag{33}$$

Thus an approximation for t may be obtained in the neighbourhood of a singular Mach line by inserting the singular portions of $\mathscr{L}$ which are displayed in table 1, into equation (33).

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

While this method is satisfactory when it is permissible to use the asymptotic expansions of the Legendre and hypergeometric functions, which occur in equation (27), there is considerable complication at points near the axis for which the asymptotic expansions are not applicable. It should be noted that it is only the singular portion of the functions occurring in equation (27) which coincides with the singular portion of $\mathscr{L}$, and the addition of a continuous function to equations (27) may yield a simpler description of the singularities. Such a description can be obtained from the functions,*

$$g(\xi,\eta) = (2m_0\pi)^{-1}\int_0^L \{[\xi-(\xi+\eta)\,u]/[u(1-u)]\}^{\frac{1}{2}}\,\mathrm{d}u \tag{34a}$$

and

$$h(\xi,\eta) = (4m_0\pi)^{-1}\,(\xi+\eta)\int_0^L \{u(1-u)/[\xi-(\xi+\eta)\,u]\}^{\frac{1}{2}}\,\mathrm{d}u, \tag{34b}$$

which were obtained for the first approximation within the leading centred expansion wave. These functions satisfy the differential equations (13a) and (13b), respectively, and so, therefore, do $(g-G)$ and $(h-H)$. Moreover, on the boundary $y = \xi+\eta = 1$, $(g-G)$ and $(h-H)$ possess continuous first derivatives in $0\leqslant\xi<1$, so that $(g-G)$ and $(h-H)$ possess continuous first derivatives throughout the field when $0\leqslant\xi<1$. Since the singular portions of G_x and H_x are identical with those of g_x and h_x, respectively, in this range, it follows from equations (32b) and (33) that G and H may be replaced by g and h in equations (30) to yield a valid description of the flow within the wave fronts occurring in $0\leqslant x<2$. The symmetric and antisymmetric properties of the singularities, which are apparent from table 1, enable equations (34) to be applied to describe the singularities for all positive x. However, as will be shown in the next section, a shock wave forms within the wave front surrounding $\eta = 0$. The work of Mahony (1956) shows that an adequate treatment of the reflexion of such a shock wave from the jet boundary requires a reformulation of the continuous flow problem downstream of this point. Because equations (13) are linear this reformulation will involve only the wave fronts as far as the first-order approximation is concerned, so that the general body of the flow is still correctly described by the conventional solution of linearized theory.

4·3. *Properties of the flow within the jet*

A complete description has now been obtained for the flow in the first portion of the jet, and it is of interest to examine the properties of this solution. It has already been observed that this solution coincides with the solution of linearized theory away from the wave fronts, so that it is the behaviour of the solution within the wave fronts which will be considered. If the previous results are collected it is found that the distribution of Prandtl angle along the axis is given by

$$t = t_0 \quad (\xi\leqslant 0), \tag{35a}$$

$$t = t_0+\xi^{\frac{1}{2}}/(2m_0) \quad (0\leqslant\xi\leqslant 4m_0\delta), \tag{35b}$$

$$t = t_0+[\xi^{\frac{1}{2}}-(\xi-4m_0\delta)^{\frac{1}{2}}]/(2m_0) \quad (\xi\geqslant 4m_0\delta,\ \xi = O(\delta)), \tag{35c}$$

$$t = t_0+[8/(2\pi\mathrm{i})]\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} p^{-1}\,\mathrm{e}^{p(1+\xi)}/I_0(p)\,\mathrm{d}p \quad (\xi\gg\delta,\ 1-\xi\gg\delta). \tag{35d}$$

* These functions are readily expressible as complete elliptic integrals for which tables are available, e.g. Jahnke & Emde (1938), and hence are especially suited for numerical work.

It is of interest to note that equations ($35c$) and ($35d$) dovetail together within the accuracy of the approximation. Thus if equation ($35c$) is approximated for ξ large in comparison with δ the equation

$$t = t_0 + \delta\xi^{-\frac{1}{2}}[1 + O(\delta/\xi)]$$

is obtained, whilst if equation ($35d$) is approximated for ξ small the resultant equation is

$$t = t_0 + \delta\xi^{-\frac{1}{2}}[1 + O(\xi)].$$

As the two leading terms are the same, there is a smooth transformation from one form to the other as ξ increases. This dovetailing of the solutions is illustrated in figure 3, where the axial distribution of Prandtl angle t, as given by the above formulae, is plotted for $\delta = 10^{-3}$ and the cases when the free stream Mach number is 2 and 5.

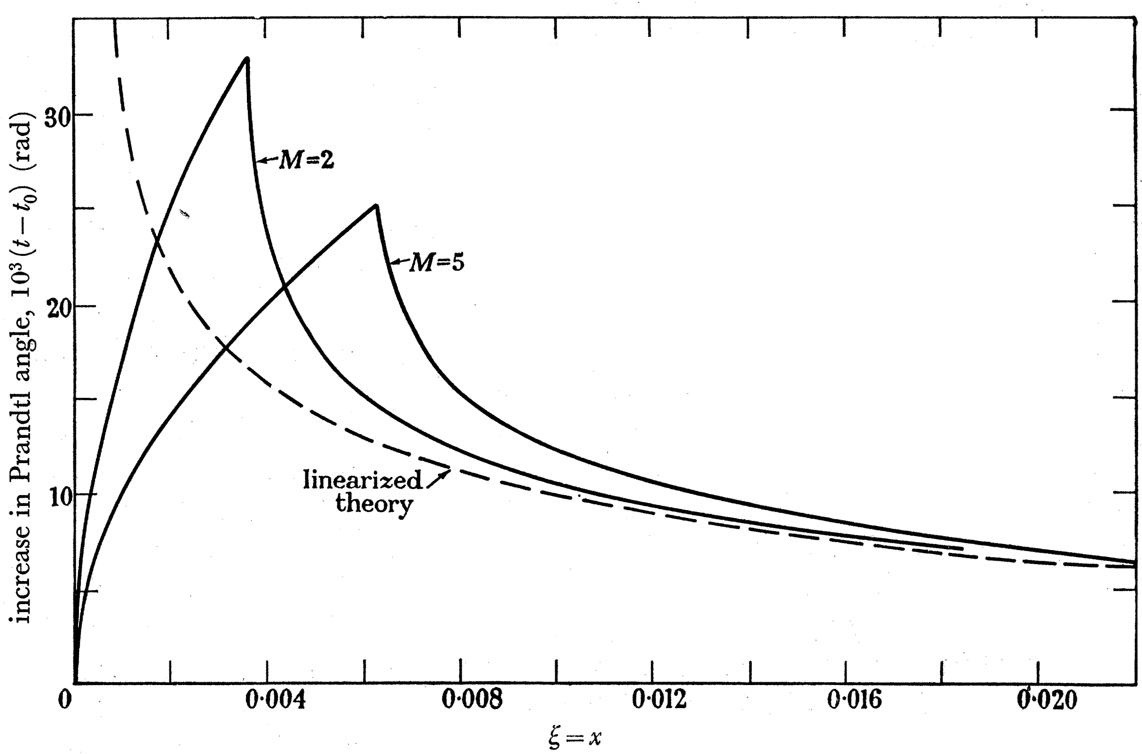

FIGURE 3. Velocity distribution along axis when $\delta = 10^{-3}$.

The greatest increase of Prandtl angle, which occurs on the axis at the point O (figure 2), is $(m_0\delta)^{\frac{1}{2}}$. This differs in order of magnitude from the increase $O(\delta)$ which occurs in the general body of the flow, so that this considerable over-expansion must be followed by a strong recompression with the possibility that a shock wave forms within the flow. Chen's results show that there can be no continuous solution of the full equations in the neighbourhood of the axis for a velocity distribution on the axis corresponding to equation ($35c$). It will now be shown that the first-order approximate solution obtained above also predicts a multi-valued velocity field in the flow plane in the neighbourhood of the Mach line DE (figure 2), and hence the existence of a shock wave in the corresponding real flow. To obtain the details of this breakdown it is necesary to ascertain the behaviour of the characteristic length parameter V within the reflected Mach wave. For illustrative purposes it is simpler to consider regions of the flow away from the axis, where the elliptic integrals in equations

(34) may be approximated by the leading term in their asymptotic expansions. A complete treatment may be applied for points near the axis at the expense of more elaborate algebra, but this only confirms the essential details deduced by the following simplified but non-rigorous analysis. At points in the reflected wave front, not too close to the axis, the value t may be shown to be*

$$t = t_0 - [(\eta + 4m_0\delta)\ln|\eta + 4m_0\delta| - \eta\ln|\eta|]/(4m_0 y^{\frac{1}{2}}) \tag{36}$$

for values of η which are $O(\delta)$. The terms $O(\delta)$ may be obtained from a careful expansion of equations (30) and (31). For any fixed y large in comparison with δ, the maximum change in Prandtl angle within the reflected wave is given by

$$t = t_0 - y^{-\frac{1}{2}}\delta\ln\delta + O(\delta),$$

and this change is in the sense of an expansion. Note that once more this solution (36) dovetails with the singular solution of linearized theory as may easily be seen by expanding equation (36) for η large in comparison with δ.† An expression similar to equation (36) may be obtained for θ, and hence V, which is determined from equation (11*b*), may be shown to be

$$V = V_0\{1 - y^{\frac{1}{2}}\ln|\eta/(\eta + 4m_0\delta)|\}.$$

Thus, on every line y constant, save the axis, V changes sign twice—at points on either side of the 'minus' Mach line $\eta = -4m_0\delta$. A shock will therefore start with zero strength on the axis from the point $\eta = -4m_0\delta$. This shock wave could be determined by methods similar to those used by Mahony (1956) in the analogous problem of two-dimensional flow. However, this lies outside the scope of the present investigation.

Before the present example is left, it is of interest to confirm that Chen's result that the derivatives of the velocity components are continuous holds away from the neighbourhood of the axis. Equation (36) shows that t_η becomes infinite on the Mach lines $\eta = 0$ and $\eta = -4m_0\delta$. The latter line is within the region of the characteristic plane which is multiply mapped in the flow plane and which is excluded from the real flow when a shock wave is introduced (see Mahony 1956). However, the line $\eta = 0$ will be associated with a Mach line in the real flow at least for some distance from the axis.‡ When the derivative with respect to the spatial co-ordinate normal to the Mach line is computed with the use of equations (5), it is found that the singularity of t_η or θ_η is cancelled by the logarithmic singularity in V, so that the spatial velocity derivative is bounded. Since singularities are propagated along Mach lines in an unchanged form subject only to continuous changes in scale, it is apparent from the form of the fundamental equations that any singularities of ξ derivatives are also associated with U save, of course, on the axis. A similar result holds for η derivatives and V. Thus the flow field with the axis excluded will always be such that the spatial derivatives of the velocity components will be bounded at all points where the velocity components are continuous.

* E.g. by applying equation (33) to the singularities of $\mathscr{L}$ displayed in table 1 or by using the asymptotic expansions of equations (34) in equations (30).

† If this dovetailing has been assumed the method, which Van Dyke (1954) and Wood (1956) used to remove an analogous singularity from the linearized potential flow around a thin wing, could have been adopted to treat the singular wave fronts. The present results, therefore, increase the likelihood that their method has general applicability in the removal of artificial singularities from approximate solutions.

‡ As the greatest change of t for a given y occurs on this line and is in the sense of an expansion it appears probable that the entire segment of the line $\eta = 0$ with $0 \leqslant \xi \leqslant 1$ lies in the real flow.

5. Conclusion

The results of the last section although obtained only for the special case when the flow upstream of the wave front is uniform, may be applied to the treatment of any problem in which the boundary conditions involve an expansive discontinuity of slope or pressure at the external boundary For the equations governing the approximate solution are essentially linear, so that superposition of solutions is valid provided wave fronts of the separate solutions do not coincide. Thus each wave front may be treated in isolation as in § 4 or in the work of Mahony (1957), and the remainder of the flow can then be determined by conventional linearized theory. Thus, it is a general result that no continuous supersonic flow is possible within a circular duct if the walls of the duct possess a discontinuity of slope.

The properties of the solution obtained in § 4 which have now been shown to be of general validity, will now be used in a *a posteriori* verification of the approximations made in § 2 as well as a guide to the order of the differences between the approximate solution and the solution of the full equations. This is most readily performed by substituting the approximate solution in the right-hand side of equations (8), and noting that the largest terms which occur are just those which have been retained in the approximate equations. The next largest terms establish the error to be anticipated in applying the approximate solution. It is apparent that the error terms are not of the same order of smallness throughout the field. Thus, within the leading wave front away from the axis the error in θ or t is $O(\delta^2)$, while near the axis the error is $O(\delta)$, while downstream of the reflected wave the error is $O(\delta^2 \ln \delta)$. However, all these terms are small in comparison with the local values of θ and t, so that the above solution does provide an approximation which may be expected to be valid everywhere.

In conclusion the author wishes to express his gratitude to Professor T. M. Cherry, F.R.S., for much helpful criticism during the preparation of this paper.

References

Chen, Y. W. 1952 *Commun. Pure Appl. Math.* **5**, 57.

Chen, Y. W. 1953 *Commun. Pure Appl. Math.* **6**, 179.

Erdelyi *et al.* 1953 *Higher transcendental functions*, vol. 1. New York: McGraw-Hill.

Erdelyi *et al.* 1954 *Tables of integral transforms*, vol. 1. New York: McGraw-Hill.

Howarth, L. (ed.) 1953 *Modern developments in fluid dynamics*, vol. 1. *High speed flow*, 1. Oxford University Press.

Jahnke, E. & Emde, F. 1938 *Tables of functions*. Leipzig: Teubner.

Mahony, J. J. 1956 *Phil. Trans.* A, **248**, 499.

Mahony, J. J. 1957 *A.R.L. Aero. Note* A, 160.

Mahony, J. J. & Meyer, R. E. 1956 *Phil. Trans.* A, **248**, 467.

Meyer, R. E. 1948 *Quart. J. Mech.* **1**, 451.

Meyer, R. E. 1949 *Phil. Trans.* A, **242**, 153.

von Kármán, Th. & Moore, N. B. 1932 *Trans. Amer. Soc. Mech. Engng*, **54**, 303.

Van Dyke, M. D. 1954 *N.A.C.A. Tech. Note*, 3343.

Ward, G. N. 1948 *Quart. J. Mech.* **1**, 225.

Ward, G. N. 1949 *Quart. J. Mech.* **2**, 75.

Watson, G. N. 1944 *Theory of Bessel functions* (2nd ed.). Cambridge University Press.

Whitham, G. B. 1950 *Proc. Roy. Soc.* A, **201**, 89.

Whitham, G. B. 1952 *Commun. Pure Appl. Math.* **5**, 301.

Wood, W. W. 1956 Ph.D. Thesis, University of Cambridge.